Cayley's theorem

Thm: If G is a group then it is isomorphic to a subgroup of S_G.

Pf: $\forall g \in G$ define $\tau_g : G \to G$ by $\tau_g(h) = gh$.

Claim: $\tau_g \in S_G$. ✓

Pf. of claim:

- injective ✓

  If $h_1, h_2 \in G$ and $\tau_g(h_1) = \tau_g(h_2)$ then $gh_1 = gh_2 \Rightarrow h_1 = h_2$.

- surjective ✓

  Suppose $k \in G$, let $h = g^{-1}k$. Then $\tau_g(h) = gh = g(g^{-1}k) = k$.

Therefore, τ_g is a bijection from G to G. $\blacksquare$

Now define $\phi : G \to S_G$ by $\phi(g) = \tau_g$. Then:

- ϕ is a homomorphism: ✓

  $\forall g_1, g_2 \in G$, $\forall h \in G$,

$$
\begin{aligned}
\phi(g_1g_2)(h) &= \tau_{g_1g_2}(h) && \text{(def. of } \phi\text{)} \\
&= (g_1g_2)h && \text{(def. of } \tau_{g_1g_2}\text{)} \\
&= g_1(g_2h) \\
&= (\tau_{g_1} \circ \tau_{g_2})(h) && \text{(defs. of } \tau_{g_1} \text{ and } \tau_{g_2}\text{)} \\
&= (\phi(g_1) \circ \phi(g_2))(h) && \text{(def. of } \phi\text{)}
\end{aligned}
$$

$\Rightarrow \phi(g_1g_2) = \phi(g_1) \circ \phi(g_2)$. (bin. op. on S_G)

• ϕ is injective: ✓

Suppose $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2)$.

Then $\tau_{g_1} = \tau_{g_2} \Rightarrow \tau_{g_1}(e) = \tau_{g_2}(e) \Rightarrow g_1 e = g_2 e \Rightarrow g_1 = g_2$.

It follows that $\phi(G) \leq S_G$, and that the map $\tilde{\phi} : G \to \phi(G)$

defined by $\tilde{\phi}(g) = \phi(g)$ is an isomorphism.

Therefore $G \cong \phi(G)$. ∎

" $\tilde{\phi}(G)$